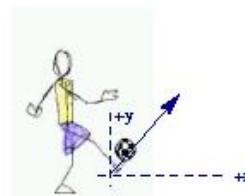


## Physics 1012 Lesson 7, Describing Motion in Two Dimensions\*

Motion is not limited to one dimension as we have previously described. Throwing balls and shooting hunting rifles are two examples. Being able to predict the trajectory of the projectile can be important. In this lesson we develop the concepts and mathematics associated with projectile motion.

A ball rolling along a flat surface moves in one dimension (horizontal or  $x$  – *direction*) as does a ball falling to the ground when dropped (vertical or  $y$ –*direction*). When throwing a baseball or kicking a soccer ball, the ball moves in a direction that is the combination of the  $x$  and  $y$  directions.

Figure I: Motion in Two Dimensions



Suppose you were asked to predict how far away the kicked soccer ball would land from the kicker. It should be apparent that this would depend on two things: 1) how hard the ball is kicked (e.g. its initial speed) and 2) the initial angle that the ball is launched; if somehow the ball were kicked straight up (an angle of  $90^\circ$  relative to the horizontal direction), the ball would fall back to the point initially kicked. If kicked at a very small angle relative to the horizontal direction the ball would land nearby. The range of the kicked soccer ball could also depend on weather conditions, e.g. wind, rain. Typically we make the assumption that our experiments are performed in perfect conditions, free from any natural impedance. Before we can solve this problem we have to go over a few basics.

Together the initial speed and direction of the ball is formally known as velocity. We've used the term velocity before ( $\bar{v} = \frac{\Delta x}{t}$ ) but did not indicate that the direction of the object was important. For now on think of velocity ( $\vec{v}$ ) as a quantity with both magnitude and direction. When both magnitude and direction are needed to fully describe a quantity it is known as a vector. Velocity (versus speed), acceleration as is displacement (as compared to distance) are vector quantities. On the other hand, time is a quantity where only magnitude is important - can you travel backwards in time? Quantities which require only the magnitude to be fully described are known as scalars.

The mathematics associated with two dimensional motion problems often requires a detailed use of **vectors**. Graphically, vectors are represented as directed line segments, e.g. lines with arrows where the length of the line is proportional to the magnitude of the vector, and the arrow indicates the direction. In Figure I the initial velocity vector is represented graphically.

An interesting and useful feature of motion in two dimensions is that the motion can be analyzed as though it were in two independent directions, e.g. horizontal ( $x$ ) and vertical ( $y$ ). This requires that the motion be separated into the horizontal and vertical components; e.g.  $x$ -component and  $y$ -component of velocity.

\* Study Chs. 3 and complete the corresponding computer pre-lesson test before working through this lesson.

**Vectors:** A vector is used to represent both magnitude and direction. Displacement, velocity and acceleration are all examples of a vector quantity. Time is an example of a non-vector or scalar quantity; with time, direction can only be positive (going back in time is infeasible) and thus only its magnitude is considered. In the example of kicking a ball, your kick results in an initial speed, for example  $20 \text{ m/s}$  and a direction, e.g. horizontally, at an angle  $\theta = 45^\circ$  or possibly straight up. The associated vector diagram might look like Figure II where components  $v_x$  and  $v_y$  are on display.

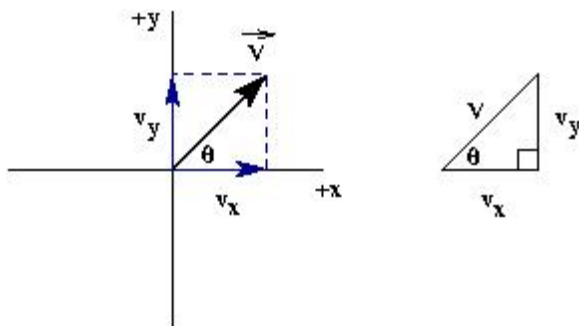


Figure II: Vector Components

Recall from the trigonometry of right triangles:  $\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$  and  $\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}}$ . Using these along with a little algebra, the  $x$  and  $y$ -components of velocity are  $v_x = v\cos\theta$  and  $v_y = v\sin\theta$  respectively.

- ? Draw a 2D orthogonal coordinate system, label the horizontal axes  $+x$ , and the vertical axis  $+y$ . Using a protractor and ruler, draw a vector  $\vec{v}$  at an angle of  $30^\circ$  relative to the horizontal axis of length of  $10 \text{ cm}$ . Determine the fraction of the vector  $\vec{v}$  that is projected onto the  $x$ -axis (a.k.a.  $x$ -component of  $\vec{v}$  or  $v_x$ ) and the fraction of the of the vector  $\vec{v}$  that is projected onto the  $y$ -axis (a.k.a.  $y$ -component of  $\vec{v}$  or  $v_y$ ) by first drawing the  $x$  and  $y$  components of the vector ( $v_x$  &  $v_y$ ) and then measuring each using a ruler. Compare your results to  $v_x = v\cos\theta$  and  $v_y = v\sin\theta$  respectively where  $v = 10 \text{ cm}$  and  $\theta = 30^\circ$ . Repeat this procedure for  $\theta = 45^\circ$  and  $\theta = 60^\circ$ . Also verify that  $v^2 = v_x^2 + v_y^2$  where  $v = 10 \text{ cm}$ .

**Solved Problem 1:** A soccer ball is kicked with a velocity of  $15 \text{ ft/s}$  and at an angle of  $45^\circ$  relative to the horizontal direction. Separate the vector which represents the initial velocity into  $x$  and  $y$ -components.

**Question:** Find  $x$  &  $y$ -components of velocity **Known:** velocity  $v = 15 \text{ ft/s}$ , angle  $\theta = 45^\circ$ .

**Principle:** Trigonometry of right triangles **Approach:** use the definitions of  $\cos\theta$  and  $\sin\theta$

The component of the velocity in the  $x$ -direction is  $v_x = 15 \cos 45 = 10.61 \text{ ft/s}$  and the  $y$ -component is  $v_y = 15 \sin 45 = 10.61 \text{ ft/s}$ . Notice that  $\cos\theta = \sin\theta = \frac{\sqrt{2}}{2} = 0.707$  when  $\theta = 45^\circ$ . Moreover  $v$ ,  $v_x$  and  $v_y$  are related as  $v = \sqrt{v_x^2 + v_y^2} = \sqrt{10.61^2 + 10.61^2} = 15 \text{ ft/s}$  due to the Pythagoras (Pythagorean Theorem).

**Motion in 2D** We learned in the previous lesson that when dropping a ball from rest  $v_1 = v_{1y} = 0$ , the speed of the ball increases by roughly  $10 \text{ m/s}$  every second, or more precisely the acceleration of a freely falling object is  $g = 9.8 \text{ m/s}^2$ . After you kick or throw a ball in two dimensions the object still experiences an acceleration of  $g$  that is directed downwards but no horizontal acceleration ( $a_x = 0$ ). Figure III illustrates

this point. Notice that neither the length or direction of  $v_x$  at each point does not change ( $a_x = 0$ ) but the vertical component ( $v_y$ ) does change ( $a_y \neq 0$ ).

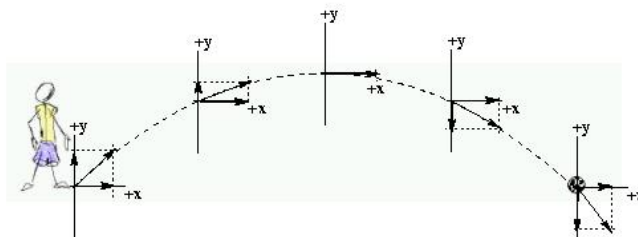


Figure III: Projectile Motion Vector Components

It is the fact that  $a_x = 0$  and  $a_y = g$  that is responsible for the parabolic trajectory associated with projectile motion. Generally, the mathematics used in projectile motion problems rely on two equations:

$$\Delta x = v_{1x}t + \frac{1}{2}a_x t^2 \quad (1)$$

$$\Delta y = v_{1y}t + \frac{1}{2}a_y t^2 \quad (2)$$

where  $v_{1x} = v_1 \cos\theta$ ,  $v_{1y} = v_1 \sin\theta$ ,  $\theta$  is the angle between the horizontal axis and the vector,  $a_x = 0$ , and  $a_y = g = -9.8 \text{ m/s}^2$  since we have chosen the convention that up is positive. With  $a_x = 0$ ,  $a_y = -g$  (since it points downwards which is in the negative  $y$ -direction), and the definitions of  $v_{1x}$  &  $v_{1y}$ , eqs. 1 & 2 become:

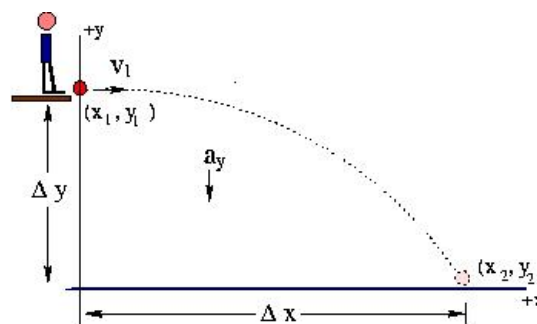
$$\Delta x = v_1 \cos\theta t \quad (3)$$

$$\Delta y = v_1 \sin\theta t - \frac{1}{2}gt^2 \quad (4)$$

Using these we can solve a variety of interesting projectile motion problems.

**Solved Problem 2:** While standing on a high bridge, a ball is kicked horizontally with a speed of  $12 \text{ m/s}$ . The time between the kick and striking the water below is  $2.1 \text{ s}$ . How high is the bridge and how far from the base of the bridge does the ball land ( $\Delta x$ )?

To solve this problem it is imperative draw a picture which includes a coordinate system that indicates the initial position  $(x_1, y_1)$  and final position  $(x_2, y_2)$ , the initial velocity of the ball (magnitude and direction), and the accelerations ( $a_x = 0$  and  $a_y$ ). Here the trajectory the ball between  $(x_1, y_1)$  and  $(x_2, y_2)$  is also shown.



**Question:** Determine the height of the bridge,  $\Delta y$ , using eq. 4. Since the initial velocity  $v_1 = 12 \text{ m/s}$  is directed exclusively in the horizontal direction ( $\theta = 0^\circ$ ), the  $y$ -component of the initial velocity,  $v_{1y} = v_1 \sin\theta = 12 \times \sin 0 = 0$ . With the time of flight  $t = 2.1 \text{ s}$  then

$$\Delta y = v_1 \sin 0t - \frac{1}{2}gt^2 = -\frac{1}{2}g \times 2.1^2 = 21.61 \text{ m}$$

Notice that a ball dropped from rest ( $v_1 = 0$ ) from a height of 21.61 m would also strike the ground in  $t = 2.1$  s. Since the horizontal component of the acceleration  $a_x$  is zero, dropping and throwing a ball in the horizontal direction result in identical fall times. Finally, the horizontal distance is  $\Delta x = v_1 \cos \theta t = 12 \text{ m/s} \times \cos 0 \times 2.1 \text{ s} = 25.2 \text{ m}$ .

The homework or test will no doubt require that you determine the horizontal distance that a ball will go, given the initial velocity, both magnitude and direction. For the case when the projectile vertical displacement ( $\Delta y$ ) is zero, as in the case of a soccer ball being kicked along a level playing field, eq. 3 & 4 can be combined:

$$\Delta y = 0 = v_1 \sin \theta t - \frac{1}{2}gt^2 \longrightarrow v_1 \sin \theta t = \frac{1}{2}gt^2 \longrightarrow v_1 \sin \theta = \frac{1}{2}g \longrightarrow t = \frac{2v_1 \sin \theta}{g}$$

Inserting this into eq. 3, gives:

$$\Delta x = v_1 \cos \theta t = v_1 \cos \theta \left( \frac{2v_1 \sin \theta}{g} \right) = \frac{2v_1^2 \cos \theta \sin \theta}{g} \quad (\text{applies only when } \Delta y = 0) \quad (5)$$

- ? A soccer ball is kicked with a velocity of 15 ft/s and at an angle of  $45^\circ$  relative to the horizontal direction. Determine the horizontal range  $\Delta x$ . Ans:  $\frac{225}{32}$  ft

**Solved Problem 3:** Angel Falls in southeastern Venezuela is the highest uninterrupted waterfall in the world, dropping 979 m (3212 ft). Using the free-fall equation from Chapter 1, it takes 14 s for the water to fall to the river below. If the water lands 50 m from the base of the vertical cliff, what was its horizontal speed at the top?

- ? Make a sketch of this situation, identify known quantities and a coordinate system.

**Question:** Find initial horizontal speed  $v_1$  **Known:**  $\Delta y = 979$  m, flight time  $t = 14$  s, horizontal distance  $\Delta x = 50$  m, acceleration due to gravity  $a = 9.8 \text{ m/s}^2$ .

Additionally, we will choose to assume that initially, there is no vertical component to the velocity,  $v_{1y} = 0$  and thus  $v_{1x} = v_1$ , that is, all of the initial velocity is horizontal. Imagine then that each droplet of water starts its flight through air with only a horizontal direction, in other words  $\theta = 0$  when the  $+x$  direction is chosen horizontal, and in the direction in which the water initially flows. Implicit here also is that the water is moving with a constant velocity  $v_1$ .

Given  $\Delta x$ ,  $\Delta y$ ,  $t$ ,  $a$ , and  $\theta = 0$ , how to find  $v_1$ ? In your opinion, which equations would prove most useful?

Naively, since  $\Delta x = v_1 \cos \theta t + \frac{1}{2}a_x t^2$  and with  $a_x$  (the horizontal component of the acceleration) identically zero and  $\theta = 0$ , then  $\Delta x = v_1 t$ , and then  $v_1 = \frac{\Delta x}{t} = \frac{50 \text{ m}}{14 \text{ s}} = 3 \text{ m/s}$ .

Had  $t$  not been explicitly given in the problem we may of been faced with another choice, how to find  $t$ ? Let's follow that logic for practice; given  $\Delta x$ ,  $\Delta y$ ,  $t$ ,  $a$ , and  $\theta = 0$  how to find  $t$ ? Reversing  $v_1 = \frac{\Delta x}{t}$  to  $t = \frac{\Delta x}{v_1}$  would not help because we (hypothetically) have not yet discovered  $v_1$ .

Trying instead  $\Delta y = v_1 \sin \theta t + \frac{1}{2} a_y t^2$ : let's see, know  $\Delta y$ , missing  $v_1$ , know  $\theta$ , need  $t$ , know  $a_y = g$ , need  $t$ . But fortunately  $\theta = 0$  and thus  $\sin \theta = \sin 0 = 0$ , then we have that  $\Delta y = 0 + \frac{1}{2} g t^2$  which can now be solved for  $t$ .

? For this problem, use  $\Delta y = \frac{1}{2} g t^2$  to verify that  $t = 14$  s.

? With  $v_1 = 3$  m/s but  $\theta = 30^\circ$  where does the water land? (Answer:  $t = 14.29$  s  $\rightarrow \Delta x = 37.1$  m)

**Solved Problem 4:** Suppose that you were the pilot of a hot-air balloon participating in a festival competition. Many balloons are launched and must travel over 10 miles to drop a light sand bag close to a large bullseye located in an open field. You find yourself traveling at 12 knots per hour (relative to the ground below) and fortunate in terms of your approach, because you will pass directly over the target. When you are 50 yards from the center of the target you throw the bag horizontally with an initial velocity of 15 ft/s. At what height above the ground should you throw the sand bag.

? Complete Step 1 of the problem solving strategy for this problem.

The variation in units for this problem is annoying. It is always a good idea to convert to standard units right away; here we will use of ft and s. A knot is a unit of speed equal to one nautical mile per hour and is approximately 1.15 statute miles per hour since one nautical mile = 6076 ft. Then the velocity of the balloon relative to the ground is  $12 \text{ knots} \times \left( \frac{1.15 \text{ mi/hr}}{1 \text{ knot}} \right) \left( \frac{1.47 \text{ ft/s}}{1 \text{ mi/hr}} \right) = 20.24 \text{ ft/s}$ . and of course  $50 \text{ yd} = 150 \text{ ft}$ ; also  $g = 32 \text{ ft/s}^2$ .

The velocity of the sand bag relative to the ground then is the combination of the balloon velocity and the release velocity or  $v_{1x} = 20.24 + 15 = 35.24 \text{ ft/s}$  and  $v_{1y} = 0 \text{ ft/s}$  due to the bag's initial horizontal direction. Beyond this, the problem is essentially the same as the previous problem.

**Question:** Find the height  $\Delta y$  **Known:**  $v_1 = v_{1x} = 35.24 \text{ ft/s}$  since  $\theta = 0^\circ$  and  $a_y = g$ .

$$\Delta y = v_1 \sin \theta t - \frac{1}{2} g t^2 = -\frac{1}{2} g t^2 \rightarrow t = \pm \sqrt{\frac{2 \Delta y}{g}} = \pm \sqrt{\frac{2 \times 150 \text{ ft}}{32 \frac{\text{ft}}{\text{s}^2}}} = \pm 3.1 \text{ s}$$

Then the distance that the sand bag lands relative to the balloon is:

$$\Delta x = v_1 \cos \theta t = v_1 \cos 0^\circ t = v_1 t = 35.24 \text{ ft/s} \times 3.1 \text{ s} = 109.24 \text{ ft}$$

and the distance it lands from the bullseye target is  $150 - 109.24 = 40.76 \text{ ft}$ .

Projectile motion problems generally are solved similarly. 1) Sketch the situation, 2) identify a coordinate system indicating the starting point of the projectile as  $(x_1, y_1)$  and the landing position as  $(x_2, y_2)$  3) identify the initial velocity vector including the angle  $\theta$  4) use eq. 3 & 4 to complete the solution and 5) check the sensibility of the solution. We model this approach once more for a slightly more complicated situation.

**Solved Problem 5:** A ball is kicked from a vertical height of  $5\text{ m}$  with a speed of  $10\text{ m/s}$  and at an angle of  $45^\circ$  relative to the horizontal direction. Will the ball land on a box which stands  $2\text{ m}$  tall and is located  $15\text{ m}$  away. The top of the box is round, with a diameter of  $2\text{ m}$ ?

**Question:** Find the horizontal range of the kicked ball **Known:** Initial height  $5\text{ m}$  and speed of the ball ( $10\text{ m/s}$  at an angle of  $\theta = 45^\circ$  relative to the horizontal. **Principle:** Projectile motion **Approach:** Use eq. 3 & 4 to determine the horizontal range  $\Delta x$  and compare it to  $15 \pm 2\text{ m}$  to see if it landed on container.

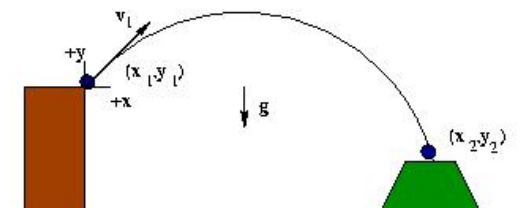


Figure IV: Problem Sketch

Eq. 3:  $\Delta x = v_1 \cos \theta t \rightarrow$  to get  $\Delta x$ , since we know  $v_1 = 10\text{ m/s}$  and  $\theta = 45^\circ$ , we need  $t$ . Use eq. 4:

Eq. 4:  $\Delta y = v_1 \sin \theta t - \frac{1}{2}gt^2 \rightarrow$  Let  $(x_1, y_1) = (0, 5)$  and  $(x_2, y_2) = (x_2, 2)$  recall that downwards is negative. Then  $\Delta y = y_2 - y_1 = 2 - 5 = -3\text{ m}$  and Eq. 4 becomes:

$$-3 = 10 \sin 45 t - \frac{1}{2} 10 t^2 \rightarrow 5 t^2 - 10 \frac{\sqrt{2}}{2} t - 3 = 0 \rightarrow 5 t^2 - 5 \sqrt{2} t - 3 = 0$$

- ? Use this equation to show that the time-of-flight is  $t = 0.34\text{ seconds}$ . Also calculate  $\Delta x$  and compare it to  $15 \pm 2\text{ m}$  to determine whether the ball makes it onto the box-top. Ans:  $\Delta x = 2.4\text{ m}$

**Summary:** For projectile motion problems, it is essential to recognize that the horizontal component of the acceleration  $a_x = 0$ . Because of this, the resulting equations become quadratic  $t$  (see eq. 3 & 4) and give reason for the parabolic trajectory.

- ? A rock is thrown horizontally from a height of  $10\text{ m}$  with a speed of  $20\text{ m/s}$ . Simultaneously an second and identical rock is dropped from rest at a height of  $10\text{ m}$ . Compare the time of flight for each rock.

In the next lesson we shift our focus from describing motion to providing an explanation of motion by discussing forces. Before moving on, let's consider one more problem as a segway into the next lesson which includes a discussion of gravity.

## Golfing on the Moon


In his famous golfing outing on the Moon in 1971, Alan B. Shepard Jr. recalled "It was a one-handed six iron because it was very clumsy with our suits," he said in 1994. "The first one I shanked. The ball came off the handle and it rolled into a crater 40 *yards* away. The next one I hit pretty flush. Here it would have gone 30 *yards*, but because there's no atmosphere there, it went about 200 *yards*."


Reference: <http://www.nytimes.com/learning/students/scienceqa/archive/960402.html>

Alan suggested a golf ball which would go 30 yards on Earth could travel 200 yards, or nearly 6 times farther on the Moon. How could this be?

What is different about the Earth and the Moon which would allow for this?

The size of the Moon in both diameter and mass is much smaller than the Earth. In effect this reduces the gravity to roughly one-sixth that of the Earth (the details are forthcoming in the next lesson).

 With  $v_1 = 53.7 \text{ ft/s}$ ,  $\theta = 45^\circ$ ,  $\Delta y = 0$  and  $a = \frac{1}{6}g_{Earth}$ , use Eq. 5 to determine  $\Delta x$  in *yards*.

 Homework corresponding to the lesson has been announced via the course web site.